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Uniform Convergence of Series

Mathematics and Physics

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UNIFORM CONVERGENCE OF SERIES

by

WILLIAM P MILLER.

THESIS

Presented for the degree of
Bachelor of Science in
MATHEMATICS and PHYSICS
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OF Bachelor of Science

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Uniform Convergence of Series

Preface

It will be our plan to treat the subject under four different sections. In the first section we will define "uniform convergence," "simple convergence," "in general uniform convergence," and "sub uniform convergence," with examples illustrating each. The second section will deal with the conditions for and the properties of uniform convergence with examples to illustrate. Section three will deal with the applications of uniform convergence. Section four will deal with the distinction between uniform convergence and the necessary and sufficient conditions for the continuity of $f(x)$, with examples. In preparing this paper I have refered to the fole

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lowing books, "Ueber den Begriff und die Anwendung des Infinitäres," Prof. E. H. Townsend. "Differential and Integral Calculus," Harnack. "Functionen Theorie," Dini and a paper by Prof. Osgood in the "Mathematical Journal," Nov. '97.

§1 Definitions. We can represent our series by

$$\sum_{n=1}^{\infty} u_n(x) = f(x) = u_1(x) + u_2(x) + \cdots\cdots + u_n(x) + \cdots\cdots,$$

where each term $u_n(x)$ is a function of (x). We will let,

$$S_n(x) = u_1(x) + u_2(x) + \cdots\cdots + u_n(x).$$

In the following discussion, we will consider $u_n(x)$ and therefore $S_n(x)$ a continous function of x.

It may be well here to define an approximation curve. Draw the plane abcd, Fig. 2, so that it is parallel to the xy plane, then the approximation curves are the curves cut out by this plane

as it moves parallel to itself and approaches the xy plane and in the limit these curves may or may not coincide with the $f(x)$.

It is necessary to make clear the difference between the simple convergence and the uniform convergence of a series. In a simple convergent series, the series converges for a constant value of x and gives the value of $f(x)$ for this particular value of x; that is, geometrically, if we take the points of the approximation curves corresponding to this constant value of x, these values have as a limiting value the corresponding value of $f(x)$ for the same value of x. This may be expressed analytically as follows. If then for any constant value of x within the interval (α, β), say $x = x_0$, the given series converges, then we must be able to find,

for an arbitrarily small positive number σ, and other positive number $\varepsilon = \frac{1}{n'}$ such that all values of δ, where $\delta = \frac{1}{n}$, are equal to or smaller in absolute value than ε, we have

(1) $$|S(\delta, x_0) - f(x_0)| < \sigma,$$

or, what is the same thing since $\delta = \frac{1}{n}$, for all values of $n > n'$, we have

(2) $$|S_n(x_0) - f(x_0)| < \sigma$$

This positive number ε, defined above as identically $\frac{1}{n'}$, varies with x and with σ. Hence for any arbitrary but constant value of σ, it must be a function of x alone. It may or may not have, when we consider its value for all x's, a lower limit greater than zero, although it must exist and be greater than zero for each constant value of x within the given interval. On the other hand the uniform convergence of the series demands that

from some definite point on all points of the approximation curves, that is the curve itself will pass over simultaneously for all values of x within the interval, into the curve represented by $f(x)$; or we can say that the series is uniformly convergent in a given interval (α, β) if for any arbitrarily small positive number δ, another positive number $\varepsilon = \frac{1}{n'}$ can be found such that, when we consider our arbitrarily chosen δ as constant the values of $\varepsilon(x)$ within the interval (α, β) will have a lower limit greater than zero, which as we have already seen is not always the case when we require simply that the series converge for each x in this interval. Let this lower limit of $\varepsilon(x)$ be ε'; then we may replace, in the case of uniform convergence, all of the ε's $[\varepsilon_1, \varepsilon_2, \varepsilon(x)]$ by ε' and the condition for uniform convergence is

$$|S(\delta,x) - f(x)| < \sigma$$

where $|\delta| \leqq \varepsilon'$, $\varepsilon' > 0$, $\alpha <$ all x's $< \beta$.

Geometrically, we can represent this as follows, in which we will represent, as is always permissible, $f(x)$ as coinciding with the x-axis.

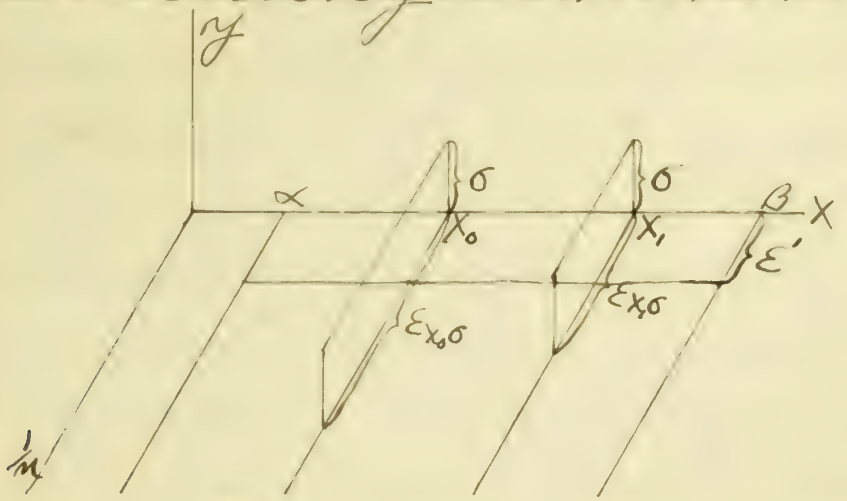


Fig. 1.

We can select σ as small as we please, but after selecting it we can let it remain constant for all values of x within (α, β). For every value of x, we have an ε depending upon x and σ, but as σ remains constant, ε will vary with x. Take any x, say x_0 then for this x we can select an ε, which we have denoted by $\varepsilon(x_0\sigma)$, such that all values of the approximation curves corre-

sponding to the value x_0, after a certain point, fall within the rectangle $(\varepsilon\, \delta)$. For every value of x in the interval we can select for n a constant ε and if these ε's have a lowest limit, say $\varepsilon' > 0$, then the series converges uniformly, because we can replace every ε by ε' and by selecting n large enough we can make every point in the approximation curves after this term, corresponding to every value of x in the interval, fall within the small region $(\varepsilon' \delta)$.

We can define uniform convergence by a geometrical definition in the following manner.

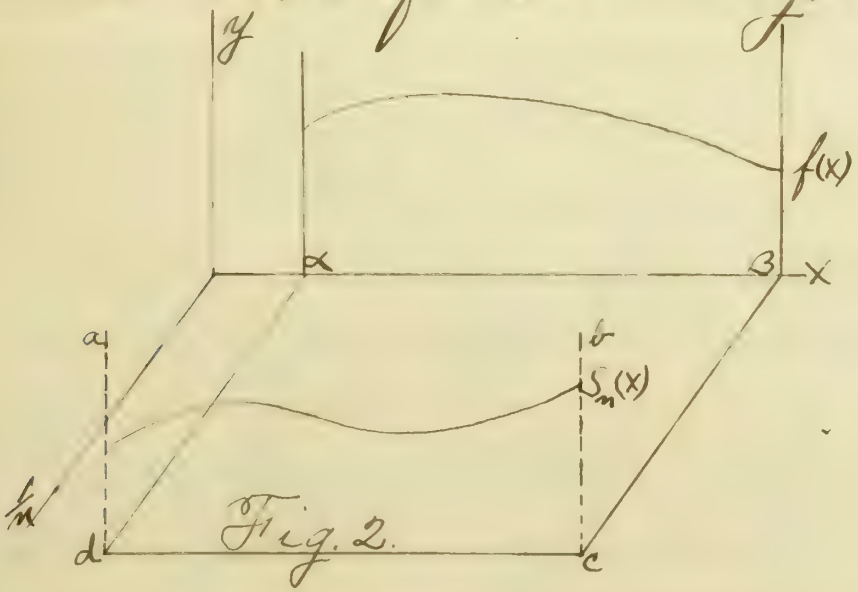


Fig. 2.

Let x, y and z be the axis in Cartesian coordinates in three dimensions and let (α, β) be an interval cut off along the x-axis and let $f(x)$ be the plot of the curve representing the series, where

$$f(x_0) = \lim_{n \to \infty} s_n(x_0) \quad \text{for } \alpha \leq x_0 \leq \beta$$

Let $s_n(x)$ represent an approximation curve. Then if the approximation curves approach the curve given by $f(x)$ simultaneously for every value of x in the interval (α, β) as the plane abcd moves parallel to and ultimately coincides with the xy-plane and if in the limit the approximation curves coincide with $f(x)$, we say that the series is uniformly convergent in the interval (α, β). We can represent this in geometry of two dimensions by projecting the approximation curves upon the xy-plane.

Let $f(x)$, fig. 3, be the curve represented by the infinite series

and let $f(x) \pm \delta$ be two curves parallel to $f(x)$ and at a distance δ from it, where δ is taken arbitrarily small and if after selecting δ, we can find an n' such that for all values of $n \geqq n'$, the projections of all the approximation curves from this point on fall within the strip $f(x) \pm \delta$, then we say that the series is uniformly convergent within the interval (α, β). We will illustrate this by the following.

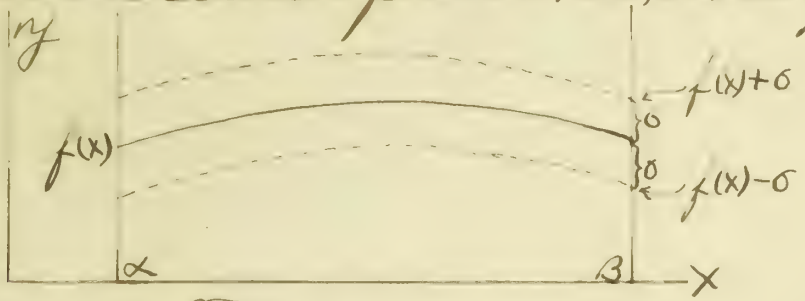


Fig. 3.

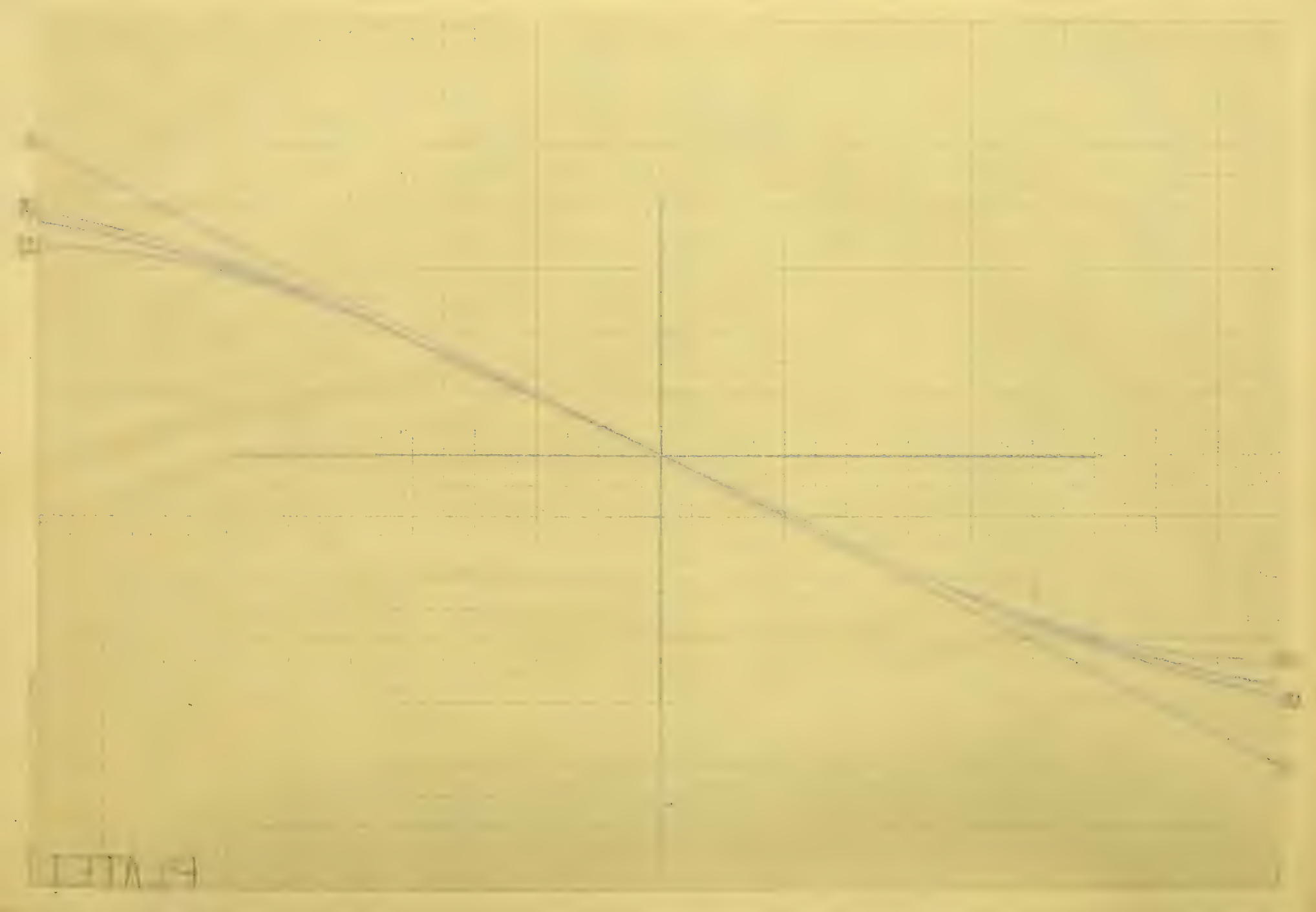
Example, Given

$$f(x) = \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \cdots\cdots$$

where $$-1 \leqq x \leqq +1$$

From the plot of this series on plate I we see that the higher the approximation curves the closer they approach to the curve representing

PLATE I



$f(X)$. Hence for all values of $n \geq n'$, where n' is sufficiently large and for an arbitrarily small, positive number σ, the approximation curves will all lie between $f(X) \pm \sigma$. This fulfills the above condition and is thus uniformly convergent.

Plot of the first approximation curve

$$y = x$$

$x = 0$	$y = 0$
$x = -1$	$y = -1$
$x = 1$	$y = 1$

Hence the first approximation curve is a straight line.

Second approximation curve.

$$y = x - \frac{x^3}{3}$$

$x = -1.0$	$y = -\frac{2}{3}$
$x = -.2$	$y = -\frac{74}{375}$
$x = -.5$	$y = -\frac{11}{24}$
$x = -.7$	$y = -\frac{1757}{3000}$
$x = 0$	$y = 0$
$x = .2$	$y = \frac{74}{375}$
$x = .5$	$y = \frac{11}{24}$

$X = .7 \qquad y = \frac{1757}{3000}$

$X = 1.0 \qquad y = \frac{2}{3}$

Third approximation curve.

$$y = x - \frac{x^3}{3} + \frac{x^5}{5}$$

$X = -1.0 \qquad y = -\frac{13}{15}$

$X = -.2 \qquad y = -\frac{4253}{46875}$

$X = -.5 \qquad y = -\frac{223}{480}$

$X = -.7 \qquad y = -\frac{428921}{1500000}$

$X = 0 \qquad y = 0$

$X = .2 \qquad y = \frac{4253}{46875}$

$X = .5 \qquad y = \frac{223}{480}$

$X = .7 \qquad y = \frac{428921}{1500000}$

$X = 1.0 \qquad y = \frac{13}{15}$

Plotting the arctan $X = f(x)$, we get

$$y = \arctan x$$

or $\tan \theta = X \qquad y = \theta = \tan^{-1} x$

$X = 0 \qquad y = 0$

$X = .087 \qquad y = 5°$

$X = .176 \qquad y = 10°$

$X = .267 \qquad y = 15°$

$X = .364 \qquad y = 20°$

$X = .466 \qquad y = 25°$

$X = .577 \qquad y = 30°$

$X = .7002 \qquad y = 35°$

$X = .834 \qquad y = 40°$

$X = 1.0 \qquad y = 45°$

In plotting these curves on the plate we have denoted the first, second, etc. approximation curves by 1, 2, 3, etc. The approximation curves will be traced in red ink and the $f(x)$ curve in blue.

We can say that a series is "in general uniformly convergent," if after cutting out a finite number of infinitely-small sub-intervals, the given series in the remainder of the given interval converges uniformly.

Example; Given

$$f(x) = \sum_{n=0}^{\infty} S_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots$$

where $$S_n(x) = \frac{nx}{1+n^2x^2}$$

and $$0 \leq x \leq 1$$

Plotting the first approximation curve as shown on plate II.

$$y = S_1(x) = \frac{x}{1+x^2}$$

$X = 0 \qquad y = 0$

$X = .1 \qquad y = \frac{10}{101} = 9.8 \text{ div.}$

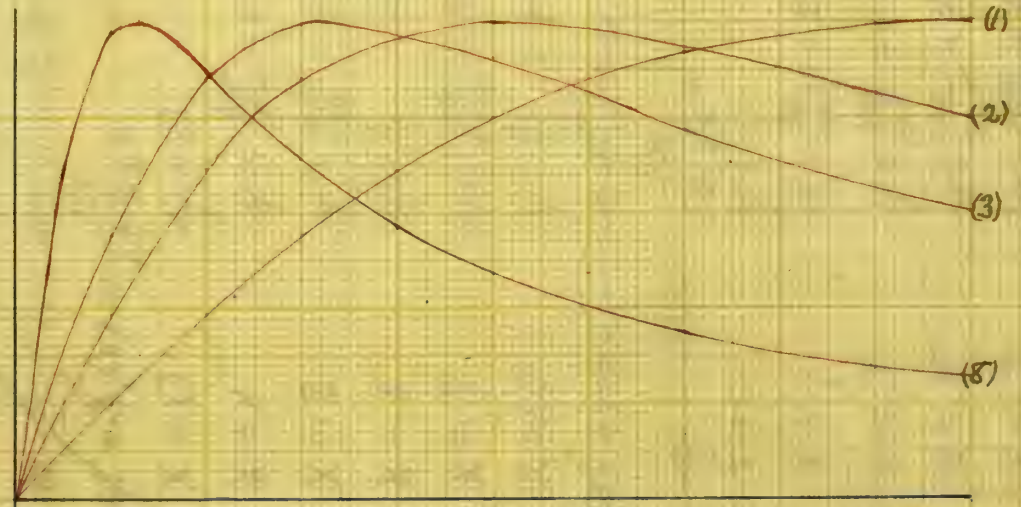


PLATE II

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$x = .2$ $\quad y = \frac{20}{.54} = 14.4$ div.

$x = .3$ $\quad y = \frac{30}{.104} = 27.6$ "

$x = .4$ $\quad y = \frac{40}{116} = 34.5$ "

$x = .5$ $\quad y = \frac{50}{125} = 40.0$ "

$x = .7$ $\quad y = \frac{70}{141} = 47.0$ "

$x = .9$ $\quad y = \frac{40}{181} = 44.4$ "

$x = 1.0$ $\quad y = \frac{1}{2} = 50.0$ "

Second approximation curve.

$$y = S_2(x) = \frac{2x}{1+4x^2}$$

$x = 0$ $\quad y = 0$

$x = .1$ $\quad y = \frac{20}{104} = 14.2$ div.

$x = .3$ $\quad y = \frac{60}{136} = 44.0$ "

$x = .5$ $\quad y = \frac{1}{2} = 50.0$ "

$x = .7$ $\quad y = \frac{140}{296} = 47.3$ "

$x = 1.0$ $\quad y = \frac{2}{5} = 40.0$ "

Third approximation curve.

$$y = S_3(x) = \frac{3x}{1+4x^2}$$

$x = 0$ $\quad y = 0$

$x = .1$ $\quad y = \frac{30}{109} = 27.6$ div.

$x = .3$ $\quad y = \frac{90}{181} = 49.8$ "

$x = .5$ $\quad y = \frac{150}{325} = 46.1$ "

$x = .7$ $\quad y = \frac{210}{541} = 38.8$ "

$x = 1.0$ $\quad y = \frac{3}{10} = 30.0$ "

Eight approximation curve.

$$y = S_8(x) = \frac{8x}{1+64x^2}$$

$X = 0$	$y = 0$
$X = .05$	$y = \frac{10}{24} = 34.4$ div.
$X = .1$	$y = \frac{20}{41} = 48.7$ "
$X = .15$	$y = \frac{30}{61} = 49.1$ "
$X = .2$	$y = \frac{40}{89} = 44.9$ "
$X = .3$	$y = \frac{60}{169} = 35.5$ "
$X = .4$	$y = \frac{80}{281} = 28.4$ "
$X = .5$	$y = \frac{4}{17} = 23.5$ "
$X = .7$	$y = \frac{140}{809} = 17.3$ "
$X = .9$	$y = \frac{180}{1321} = 13.6$ "
$X = 1.0$	$y = \frac{8}{65} = 12.3$ "

$$f(x) = \lim_{n=\infty} S_n(x) = \lim_{n=\infty} \frac{nx}{1+n^2x^2} = \lim_{n=\infty} \frac{x}{\frac{1}{n}+nx^2} = 0$$

Hence we see that $f(x)$ coincides with the x-axis. We see from plate II that the first approximation curve of this series has a maximum value of one half for $x = 1$, the second approximation curve has the maximum value one half for $x = .5$ We see that each approximation curve has a maximum value one half and the larger we take n, where n denotes the term of the series, the nearer the maximum

value as x approaches the y axis and the shape of the peak of the approximation curve becomes. If we make n large enough, this peak approaches as near as we please to the y-axis but can never reach it. Thus if we select an infinitely small region to the right of $x=0$, we can by making n large enough cause the peak to fall within this small region. Hence the approximation curves in the remainder of the interval,

$$0 < \alpha \leqq x \leqq 1$$

in the limit coincide with the original function $\varphi(x)$ and thus if we cut out the small interval $(0,x)$ in which the peak lies, the series is uniformly convergent in the remainder of the interval

$$0 < \alpha \leqq x \leqq 1.$$

Hence according to our definition we would say that this series was "in general uniformly convergent in the interval, $0 \leqq x \leqq 1$.

We say that a series is "sub uniformly convergent" in an interval, if for a certain set of n's the series converges uniformly in the given interval, but does not converge uniformly for all n's.

Example: Given the series $\sum_{n=\infty} S_n(x)$ which is formed according to the following law, namely

$$u_{2n-1}(x) = x^{n+1}, \quad u_{2n}(x) = -x^{n+1}\left(1 - \frac{1}{(n+1)!}\right)$$

where $0 \leq x \geq b < 1$.

Plotting the first approximation curve as shown in plate III.

$$y = S_1(x) = x^2$$

$x = 0$	$y = 0$
$x = .1$	$y = .01$
$x = .2$	$y = .04$
$x = .3$	$y = .09$
$x = .4$	$y = .16$
$x = .5$	$y = .25$
$x = .6$	$y = .36$
$x = .7$	$y = .49$
$x = .8$	$y = .64$
$x = .9$	$y = .81$
$x = 1.0$	$y = 1$

PLATE III

EUGENE DIETZGEN CO., CHICAGO.

Second approximation curve.

$y = S_2(x) = \frac{1}{2}x^2$

$x=0$	$y=0$
$x=.1$	$y=.005$
$x=.2$	$y=.02$
$x=.3$	$y=.045$
$x=.4$	$y=.08$
$x=.5$	$y=.125$
$x=.6$	$y=.18$
$x=.7$	$y=.245$
$x=.8$	$y=.32$
$x=.9$	$y=.405$
$x=1.0$	$y=.5$

Third approximation curve.

$y = S_3(x) = x^3 + \frac{x^2}{2}$

$x=0$	$y=0$
$x=.1$	$y=.006$
$x=.2$	$y=.028$
$x=.3$	$y=.072$
$x=.4$	$y=.144$
$x=.5$	$y=.250$
$x=.6$	$y=.396$
$x=.7$	$y=.588$
$x=.8$	$y=.832$
$x=.9$	$y=1.134$

$x = 1.0 \qquad y = 1.5$

Fourth approximation curve.

$$y = S_4(x) = \frac{x^2}{2} + \frac{x^3}{6}$$

$x = 0 \qquad y = 0$

$x = .1 \qquad y = .00376$

$x = .2 \qquad y = .0213$

$x = .3 \qquad y = .049$

$x = .4 \qquad y = .0907$

$x = .5 \qquad y = .146$

$x = .6 \qquad y = .196$

$x = .7 \qquad y = .302$

$x = .8 \qquad y = .405$

$x = .9 \qquad y = .5265$

$x = 1.0 \qquad y = .666$

Fifth approximation curve.

$$y = S_5(x) = x^4 + \frac{x^2}{2} + \frac{x^3}{6}$$

$x = 0 \qquad y = 0$

$x = .1 \qquad y = .00526$

$x = .2 \qquad y = .0229$

$x = .3 \qquad y = .0571$

$x = .4 \qquad y = .1163$

$x = .5 \qquad y = .2085$

$x = .6 \qquad y = .3256$

$x = .7 \qquad y = .5421$

$x = .8 \qquad y = .8146$

$x = .4 \qquad y = 1.1826$

$x = 1.0 \qquad y = 1.666$

$$f(x) = \frac{L}{n=\infty} S_n(x) = x^2 - x^2 + \frac{x^2}{2!} + x^3 - x^3 + \frac{x^3}{3!} + \cdots\cdots$$

$$= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots$$

$$= e^x - x - 1$$

Ploting this curve we have the curve marked $f(x)$.

In this we see that the even terms which are represented by

$$\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots$$

converge uniformly, because the maximum values taken absolutely

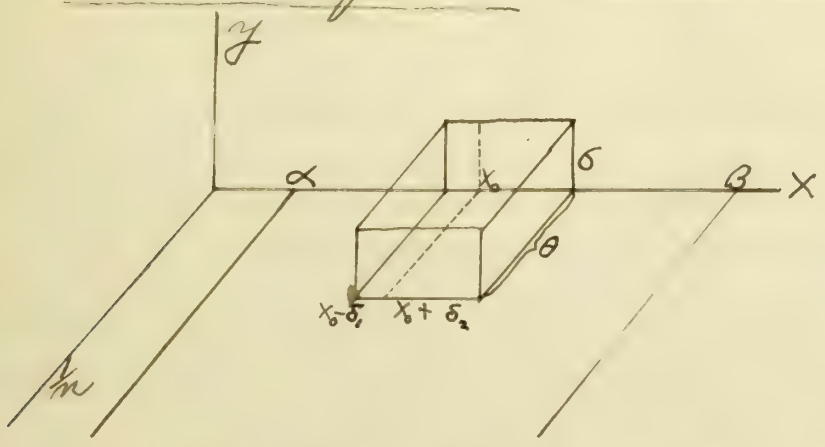
$$\sum_{k=1}^{\infty} |u_k| = \left\{\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\cdots\right\} < \sum_{1}^{\infty} \frac{1}{n^2}$$

converges, since the series $\sum \frac{1}{n^2}$ converges. But if we consider the odd terms we see that they do not converge uniformly. Hence according to our definition this series converges sub-uniformly in the interval $0 \leq x \leq 1$.

§2. *conditions for uniform convergence.* In this section, we will discuss the conditions for uniform convergence. The first condition we will give may be stated as follows:

I *The necessary and sufficient condition for uniform convergence is, that the double limit* $\lim_{\substack{n=\infty \\ x=x_0}} S_n(x)$ *exists and is equal to the value of the function for every value of* x *within the given interval, the end points included;*[1] *i, e,*

$$\lim_{\substack{x=x_0 \\ n=\infty}} S_n(x) = f(x_0); \quad \text{where } \alpha \leqq x_0 \leqq \beta$$



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(1) See, "Begriff und die Umwendung des Doppellimes," by Prof. E. J. Townsend, page 30, §13.

Proof. We will first show that our condition is necessary. We will suppose that $S_n(x)$ converges uniformly toward $f(x)$. We must then show that

$$\lim_{\substack{x=x_0 \\ n=\infty}} S_n(x) = f(x_0)$$

for every value of x_0 within the given interval, the end points included; that is we must show that around the point x_0 there must exist a definite region greater than zero, say $x_0 - \delta_1 \leqq x \leqq x_0 + \delta_2$ and $n > n' = \frac{1}{\theta}$, within which $S_n(x)$ at every point satisfies the inequality

(1) $$|S_n(x) - f(x_0)| < \sigma$$

where σ represents an arbitrarily small positive number. Since $S_n(x)$ converges uniformly toward $f(x)$, we can select a θ which may be constant for every value of x within the interval. By our supposition, $S_n(x)$ is a continuous function, thus we can say that

$$\lim_{x=x_0} S_n(x) = S_{n'}(x_0) \quad ; \quad n = \text{constant}$$

or (2) $$|S_{m'}(x) - S_{m'}(x_0)| < \frac{\sigma}{3} \; ; \; x_0 - \delta_1 \leqq x \leqq x_0 + \delta_2 \qquad m' = \frac{1}{\theta}$$

However, $S_m(x)$ converges for every x in this interval and hence we have

$$\lim_{m=\infty} S_m(x) = f(x)$$

and therefore

(3) $$|S_m(x) - S_{m'}(x)| < \frac{\sigma}{3} \; ; \; m \geqq m' = \frac{1}{\theta} \; ; \; x_0 - \delta_1 \leqq x \leqq x_0 + \delta_2$$

Then $(x_0 - \delta_1, x_0 + \delta_2, \theta)$ is the desired region about the point x_0. Since θ is constant, we can write for n sufficiently large

(4) $$|S_{m'}(x_0) - f(x_0)| < \frac{\sigma}{3}$$

Adding (2), (3) and (4) we obtain

$$|S_m(x) - f(x_0)| < \sigma$$

for every n greater than some definite value n' and for every x in the interval $(x_0 - \delta_1, x_0 + \delta_2)$. Since x was any point of the interval $(x_0 - \delta_1, x_0 + \delta_2)$, we have the double limit existing for every point in the interval and consequently our condition is the necessary condition for uniform convergence.

But this is also a sufficient condition. In order to prove this, we must show that there is a lowest θ within the given interval. As the dimensions of the small region about each point are different from zero, then for an arbitrarily small, positive σ, no point $(x, \frac{1}{n})$ in this region of x_0 can be found for which

$$|S_n(x) - f(x_0)| > \sigma$$

For if we let σ remain fixed, then the dimensions of the small region $(x_0 - \delta_1, x_0 + \delta_2, \theta)$ [i.e. $(\delta_1 + \delta_2)$ and θ] are both functions of x and may vary as x varies. These dimensions must have a lower limit greater than zero, or we would have no region satisfying the required conditions around the point, $\bar{x}_0$, where $\bar{x}_0$ is the limiting point of the x_0's as the region about x_0 approaches zero. The point $\bar{x}_0$ must also be a point of our interval (α, β), because the points in the interval form by hypothesis a perfect menge. Thus

we have shown that we have a lowest θ greater than $\tfrac{1}{2}$, no, say θ' and if we choose the same θ' for all values of x in the interval; i, e, replace all θ's by θ', then according to our definition of uniform convergence our theorm is proved.

Example: Given

$$f(x) = \lim_{n=\infty} S_n(x), \text{ where } S_n(x) = \frac{n^2 x}{1 + n^3 x^2}$$

and $0 \leqq x \leqq 1$

Let $x = \frac{1}{n}$, then we have

$$\lim_{n=\infty} \frac{n}{1+n} = \lim_{n=\infty} \frac{1}{\frac{1}{n}+1} = 1$$

Let $x = \frac{2}{n}$, then we have

$$\lim_{n=\infty} \frac{2n}{1+4n} = \lim_{n=\infty} \frac{2}{\frac{1}{n}+4} = \frac{1}{2}$$

and hence the limit

$$\lim_{\substack{n=\infty \\ x=0}} \frac{n^2 x}{1+n^3 x^2}$$

does not exist, because, if the limit $\lim_{\substack{n=\infty \\ x=0}} S_n(x)$ exists and is equal to A, then we must by every continous approach arrive at the same limiting value. Hence the series does not converge uniformly

PLATE IX

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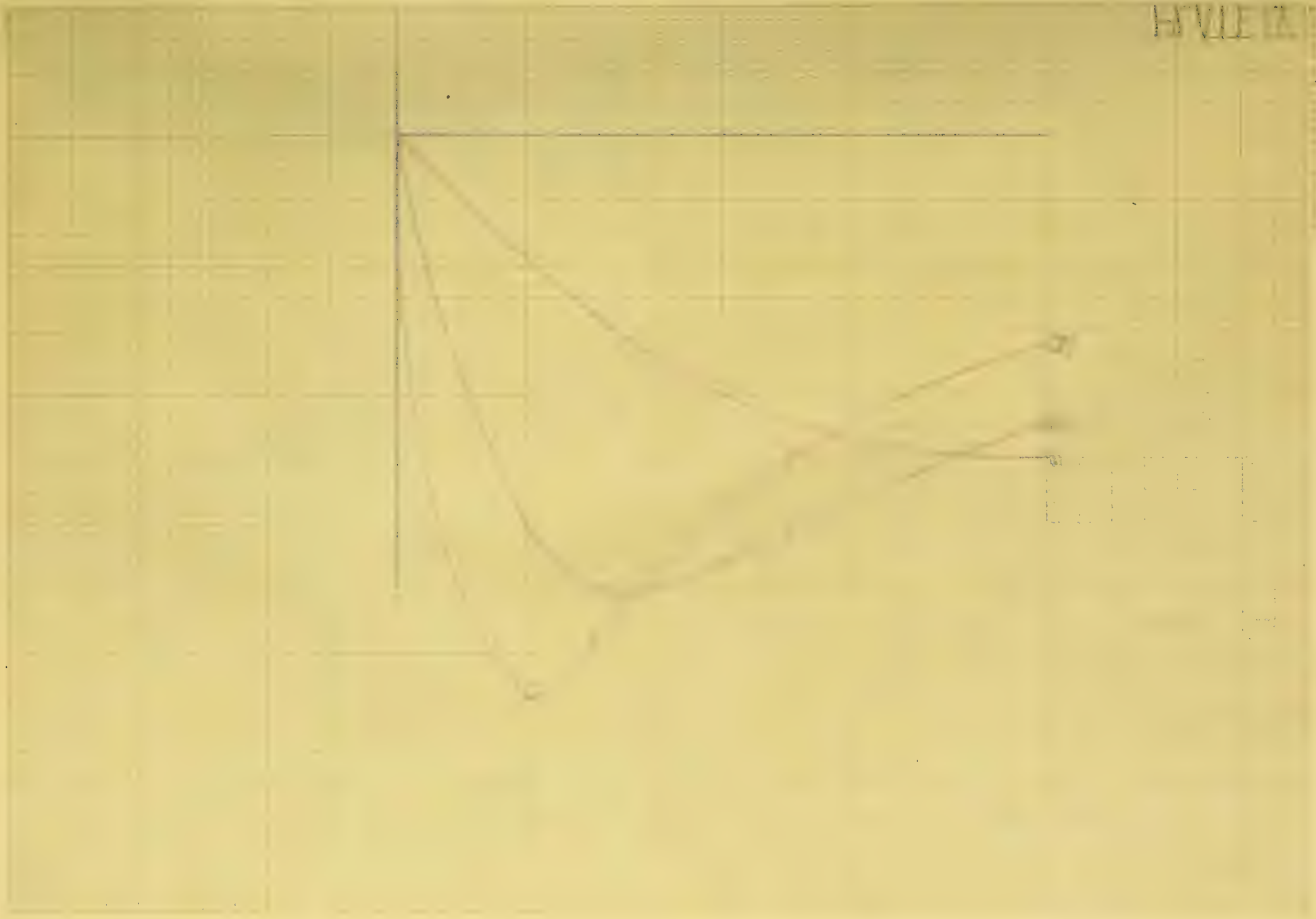


in the interval $0 \leqq x \leqq 1$. If we plot the approximation curves of $f(x)$, we get the curves represented by the pilots on plate IV. The first approximation curve is found by letting $n=1$, then we get

$$y = S_1(x) = \frac{x}{1+x^2}$$

Let $x=0$	$y=0$
$x=.1$	$y=\frac{10}{101}=9.8$ div
$x=.2$	$y=\frac{20}{104}=19.4$ "
$x=.3$	$y=\frac{30}{109}=27.6$ "
$x=.4$	$y=\frac{40}{116}=34.5$ "
$x=.5$	$y=\frac{50}{125}=40.0$ "
$x=.7$	$y=\frac{70}{149}=47.0$ "
$x=.9$	$y=\frac{90}{181}=49.4$ "
$x=1.0$	$y=\frac{1}{2}=50.0$ "

Second approximation curve.

$$y = S_2(x) = \frac{4x}{1+8x^2}$$

$x=0$	$y=0$
$x=.1$	$y=\frac{10}{27}=46.9$
$x=.2$	$y=\frac{20}{33}=60.6$
$x=.3$	$y=\frac{30}{43}=69.7$
$x=.4$	$y=\frac{40}{57}=70.2$

$X = .5$	$y = \frac{2}{3} = 66.6$
$X = .6$	$y = \frac{60}{97} = 61.7$
$X = .7$	$y = \frac{70}{123} = 56.7$
$X = .8$	$y = \frac{80}{153} = 52.3$
$X = .9$	$y = \frac{90}{187} = 48.2$
$X = 1.0$	$y = \frac{4}{9} = 44.5$

Third approximation curve

$$y = S_3(x) = \frac{7x}{1 + 27x^2}$$

$X = 0$	$y = 0$
$X = .1$	$y = \frac{70}{127} = 70.8$
$X = .2$	$y = \frac{45}{52} = 86.5$
$X = .3$	$y = \frac{270}{343} = 78.7$
$X = .4$	$y = \frac{99}{133} = 67.6$
$X = .5$	$y = \frac{18}{31} = 58.0$
$X = .6$	$y = \frac{135}{268} = 50.3$
$X = .8$	$y = \frac{180}{457} = 39.3$
$X = 1.0$	$y = \frac{9}{28} = 32.1$

$$f(x) = \sum_{n=0}^{\infty} S_n(x) = \sum_{n=0}^{\infty} \frac{n^2 x}{1 + n^3 x^2} = \sum_{n=0}^{\infty} \frac{x}{\frac{1}{n^2} + nx^2} = 0$$

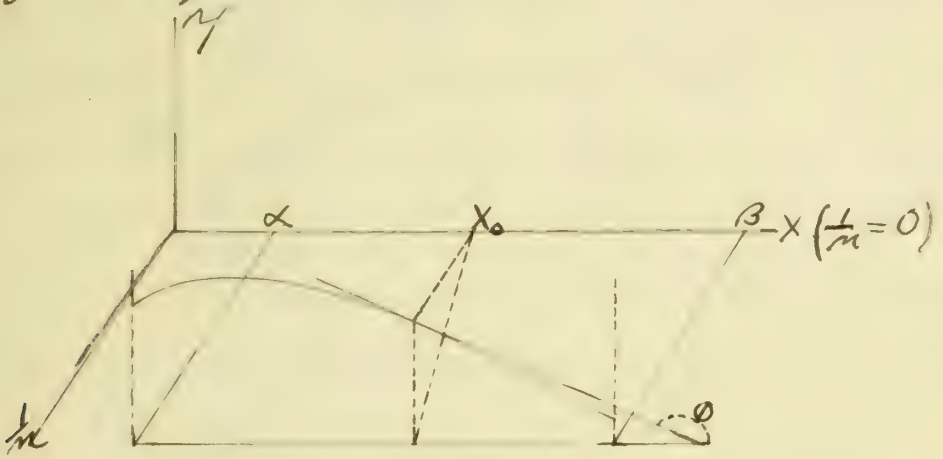
for values of x between 0 and 1. Hence $f(x)$ coincides with the x axis. We see from the plot that the approx-

imation curves have a peak which moves closer to the y axis as n approaches infinity. We also see that the peaks get sharper and higher as $n = \infty$. By making n as large as we please, we can make this peak approach as near to the y axis as we please, but it will never coincide with it; i, e, if we take a small interval to the right of $x = 0$, we can make n large enough so that the peak will fall within this small interval, thus our conclusion is verified that this series is non uniformly convergent in the interval $0 \leqq x \leqq 1$.

II If within a given interval the limit $\lim_{n=\infty} S_n(x)$ exists for a set of points every where dense, and if for all values of x within the interval and for all finite values of n, we have $|S_n'(x)| < G$, where G denotes a finite number, and

$S_n'(x)$ is the derivitive of $S_n(x)$, then the series converges uniformly in the given interval.[1]

Proof: We will consider a system of approximation curves, viz;



the curves in which the surface $z=S_n(x)$ can be cut by the planes $\frac{1}{m}=y_1, y_2, \cdots y_m$ where $y_1, y_2 \cdots y_m$ are a sequence of numbers which have the limit point zero. According to our hypothesis there exists for all x's in the interval $\alpha \leqq x \leqq \beta$, the derivitive $S_n'(x)$ and this equals

---

[1] See, "Begriff und die Umwendung des Dofpellimes," by Prof. E. J. Townsend, page 61, §14.

$\{u_1'(x) + u_2'(x) + \cdots\cdots + u_n'(x)\}$ so long, at least, as n is finite. These derivitives gives us the direction of the curve $z = S_n(x)$ for all values of x; i, e, it gives us the trigiometric tangents of the angles between the line $\frac{1}{n} = y_n$ and the tangent to the curve as shown in the figure. We will denote this angle by φ. Thus for all x's and for every constant $n = 1, 2, 3$ --- etc.

$$|S_n'(x)| = |\tan \varphi| < G.$$

On account of the continuity of $S_n(x)$ with respect to x, we can find for any point $x = x_0$ in the interval $\alpha \leqq x_0 \leqq \beta$ a small region $(x_0 - \delta_1, x_0 + \delta')$, such that upon the line $\frac{1}{n} = y$, for every x within the interval we have,

$$|S_1(x) - S_1(x_0)| < \sigma$$

To shorten our notation, we will put

$$|S_n(x) - f(x)| \equiv R_{0n}(x)$$

where $(x_0 - \delta) < x < (x_0 + \delta')$ and let σ_1, σ_2 --- --- σ_n --- be a sequence of numbers

such that $\lim_{n=\infty} \delta_n = 0$. Since

$$\lim_{n=\infty} y_n(x_0) = f(x_0)$$

for a set of constant values of x_0, every where dense within the interval (α, β), we can always find within every arbitrarily small interval at least one value of x, such that if y_n approaches the line $y_n = y$, $R_{0n}(x)$ also from a certain point on decreases for the same values of x. Let $x = \bar{x}$ be any such value within the interval $(x_0 - \delta_1, x_0 + \delta')$. But if

$$\lim_{n=\infty} R_{0n}(\bar{x}) = 0$$

with the change of y_n, then R_{0n} decreases at the same time for all values of x in the region of x; or there is a point in the same neighborhood where $|\tan \varphi|$ increases and will ultimately become greater than $\frac{1}{2}$, but by hypothesis this cannot be true. Since the points within the interval $(x_0 - \delta_1, x_0 + \delta')$ are every where dense, we can

choose a (y_1) such that $R_{o_1}(x) \leqq o_1$, where $y_0 < y_1$ and $(x_0 - \delta_1 < x < x_0 + \delta')$. In the same manner, we can choose a (y_2) such that $R_{o_2}(x) \leqq o_2$ where $y_0 < y_2 < y_1$ and $(x_0 - \delta_1) < x < (x_0 + \delta')$.

Continuing in this manner we have in general $R_{o_n}(x) \leqq o_n$, where $y_0 < y_n < y_{n-1}$ and $(x_0 - \delta_1) < x < (x_0 + \delta')$.

Thus we have when the above process is thought of as repeated an infinitely great number of times

$$\lfloor_{n=\infty} R_{o_n}(x) \leqq \lfloor_{n=\infty} o_n = 0$$

for all x's within the same interval $(x_0 - \delta_1 < x < x_0 + \delta')$. Hence within this interval $S_n(x)$ converges regularly toward $f(x_0)$. But x_0 was any point of the interval (α, β) α and β included and thus it follows, that $S_n(x)$ converges regularly toward $f(x_0)$ in the region of every point within the interval (α, β) and, therefore uniformly in the entire given interval. With this

our theorm is demonstrated.

Example: Given

$$f(x) = \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\cdots \pm \frac{x^n}{n} + \cdots\cdots;$$

where $-1 < a \leqq x \leqq +1$

The first derived function of the first n terms is

$$S_n'(x) = 1 - x + x^2 - x^3 + \cdots\cdots \pm x^{n-1}$$

The sum of the n terms of this series, when n is finite, is less than some finite number G, for all values of x between $-1 < a \leqq x \leqq 1$. Since the limit as $n = \infty$ exists for every point in the interval, but exists for a set of points everywhere dense. Hence by our previous theorm, the series representing the $\log_e(1+x)$ is uniformly convergent in the interval

$$-1 < a \leqq x \leqq 1$$

The plot of the first approximation curves are shown on plate V. The first approximation curve

$$y = x$$

represents a straight line through the origin.

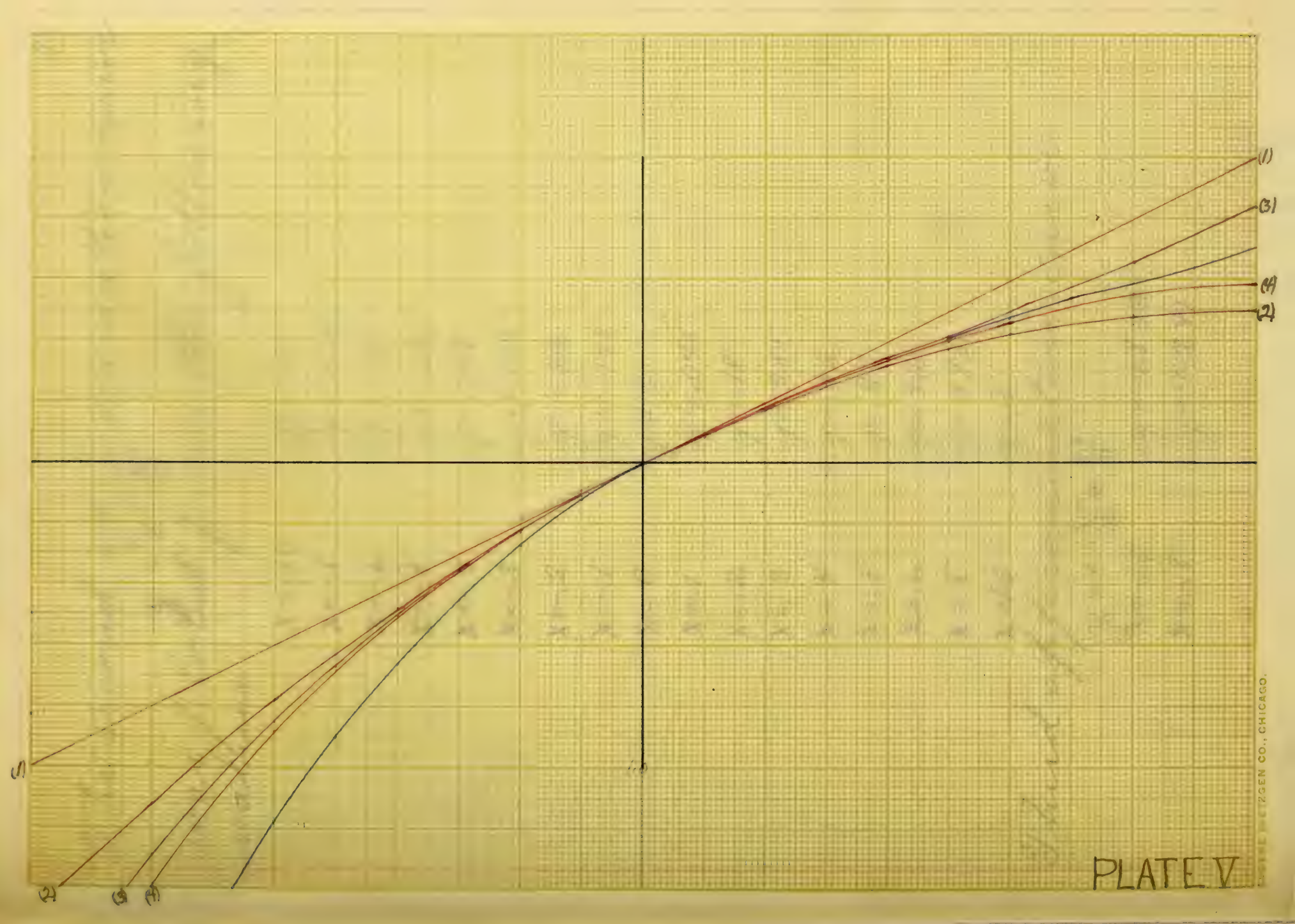


PLATE V

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PLATE 2

The second approximation curve

$$y = x - \frac{x^2}{2}$$

was plotted from the following values

$x = -1.0$	$y = -1.5$
$x = -.8$	$y = -1.12$
$x = -.6$	$y = -.78$
$x = -.5$	$y = -.625$
$x = -.4$	$y = -.48$
$x = -.3$	$y = -.345$
$x = -.2$	$y = -.22$
$x = -.1$	$y = -.105$
$x = 0$	$y = 0$
$x = .1$	$y = .095$
$x = .2$	$y = .18$
$x = .3$	$y = .255$
$x = .4$	$y = .32$
$x = .5$	$y = .37$
$x = .6$	$y = .42$
$x = .8$	$y = .475$
$x = 1.0$	$y = .5$

Third approximation curve.

$$y = x - \frac{x^2}{2} + \frac{x^3}{3}$$

$x = -1.0$	$y = -1.833$
$x = -.8$	$y = -1.291$

$x = -.6$	$y = -.852$
$x = -.5$	$y = -.667$
$x = -.4$	$y = -.501$
$x = -.3$	$y = -.354$
$x = -.2$	$y = -.223$
$x = -.1$	$y = -.1053$
$x = 0$	$y = 0$
$x = .1$	$y = .046$
$x = .2$	$y = .183$
$x = .3$	$y = .264$
$x = .4$	$y = .341$
$x = .5$	$y = .417$
$x = .6$	$y = .492$
$x = .8$	$y = .651$
$x = 1.0$	$y = .833$

Fourth approximation curve.

$$y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

$x = -1.0$	$y = -2.083$
$x = -.8$	$y = -1.3934$
$x = -.6$	$y = -.8844$
$x = -.5$	$y = -.6826$
$x = -.4$	$y = -.5074$
$x = -.3$	$y = -.356$
$x = -.2$	$y = -.2234$
$x = -.1$	$y = -.10532$

$x=0$	$y=0$
$x=.1$	$y=.0454$
$x=.2$	$y=.1814$
$x=.3$	$y=.2554$
$x=.4$	$y=.34$
$x=.5$	$y=.394$
$x=.6$	$y=.46$
$x=.8$	$y=.552$
$x=1.0$	$y=.58$

Ploting $f(x) = \log_e(1+x)$ from the table of natural logarithms, we obtain the curve drawn with blue ink and we can see from the plot that the greater we take n, the closer the approximation curve approaches $f(x)$.

The theorm which we will state next gives a sufficient, but not a necessary condition for uniform convergence.

III. If the sum of the maximum values of the seperate terms of a series converges absolutely within a given interval, then the series converges uniformly

within this interval."[1]

Proof: Let M_k be equal to the maximum value of the term $u_k(x)$ of the infinite series, then by hypothesis the series

$$\sum_{k=1}^{\infty} |M_k| ;$$

i,e, the limit $\lim_{n=\infty} \sum_{k=1}^{n} |M_k|$ exists. We can see that

$$|\text{Max. value of } S_n(x)| \leq \sum_{k=1}^{n} |M_k|,$$

because, "the absolute value of the sum of a finite number of terms can never exceed the sum of the absolute values of the individual terms." Since $\Sigma |M_k|$ converges by hypothesis,

$$|\text{max value of } S_n(x)|$$

must also converge, since it is less than or at most equal to the converging series $\Sigma |M_k|$. Hence the limit

$$\lim_{n=\infty} |\text{max. value of } S_n(x)|$$

---

[1] See, "Begriff und die Umwendung des Doppellimes," by Prof. E. J. Townsend. page 62, Theorem III.

must exist for all values of x within the interval. Let $x = x_0$ be any value of x within the interval, then around x_0 is a small region, within which for all values of x and for a sufficiently large n the function $S_n(x)$ differs from some definite number A by less than σ; i, e,

$$|A - S_n(x)| < \sigma$$

where σ is an arbitrarily small positive number; or, in other words, the limit

$$\lim_{\substack{x=x_0 \\ n=\infty}} S_n(x) = A.$$

We now have to show that the limit is equal to $f(x)$, which can be done in the following manner. Since the series of maximum values of the separate terms converges absolutely, the given series also converges for each constant value of x within the given interval. In double limits we have the following theorem:

"If the double limit $\lim_{\substack{x=a \\ y=b}} f(x,y)$ exists and is equal to A, then the single limit $\lim_{y=b} f(a,y)$ also exists and equals A." (1)

From this we have, since $\lim_{x=x_0} S_n(x) = A$ the result that

$$\lim_{\substack{x=x_0 \\ n=\infty}} S_n(x) = \lim_{n=\infty} S_n(x_0) = \lim_{n=\infty} \{u_1(x_0) + u_2(x_0) + \cdots + u_n(x_0)\} = A$$

and since this series converges for every constant x and hence for $x = x_0$, we have

$$\lim_{n=\infty} \{u_1(x_0) + u_2(x_0) + \cdots + u_n(x_0)\} = f(x_0)$$

Thus we have

$$\lim_{\substack{x=x_0 \\ n=\infty}} S_n(x) = f(x_0)$$

and, therefore our series converges regularly for every value of x in the interval, the end points

---

(1) See "Begriff und die Einwendung des d of Differentiales. By E. J. Townsend, page 10, Theorem V. Prof. Townsend discussed this theorm for the special case that $u_n(x)$ was a continous function but we have taken $u_n(x)$ as any function.

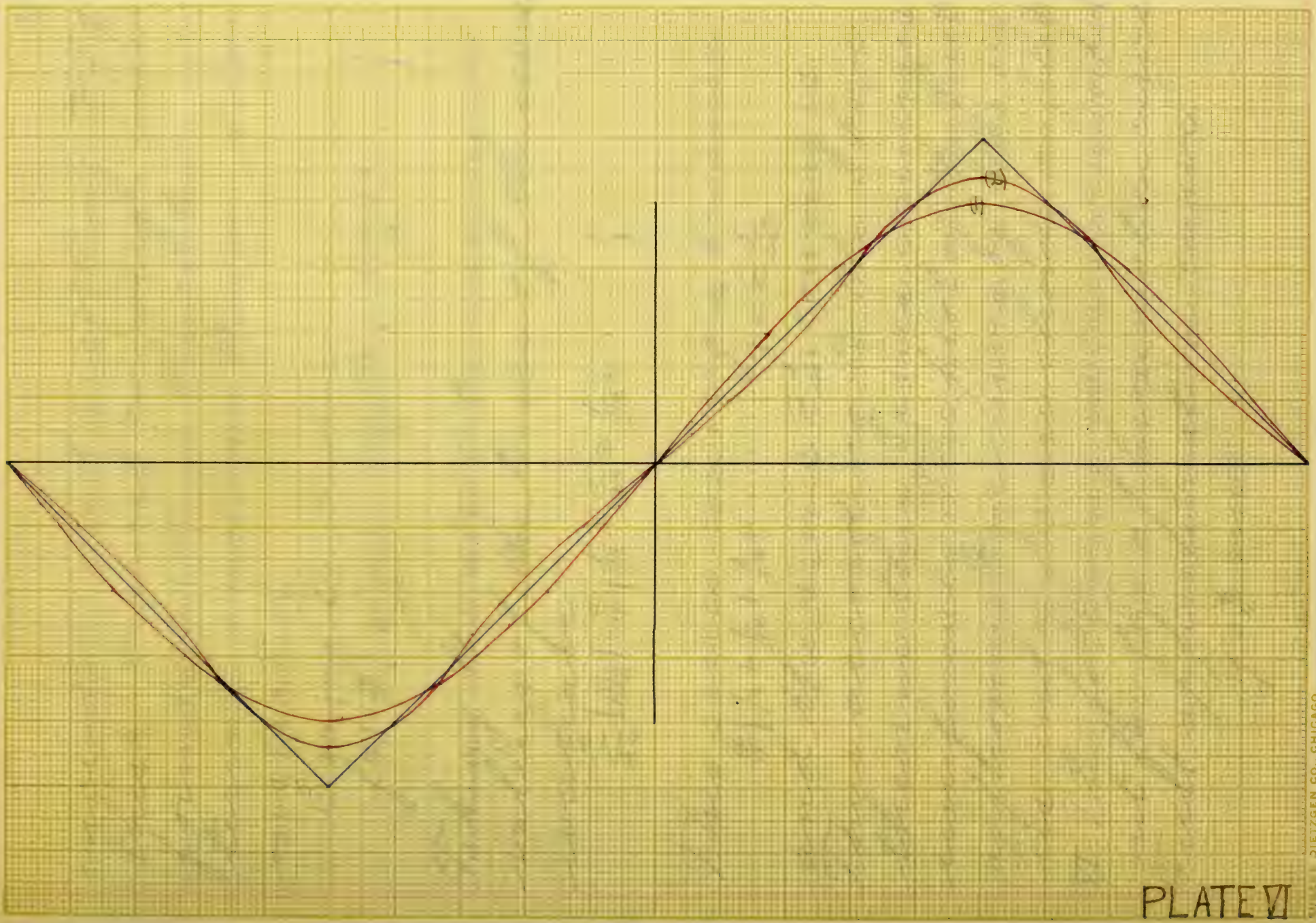


PLATE VI

DIETZGEN CO., CHICAGO.

included, and according to I of this §, it must converge uniformly within the entire interval and our theorem is demonstrated.

Example: Given

$$f(x) = \frac{4}{\pi}\left\{ \frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\cdots \right\}$$

Taking the absolute maximum values of the terms of the series, we have

$$\sum_{n=1}^{\infty} |M_n| = \frac{4}{\pi}\left\{ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots \right\}$$

This series converges, because

$$\frac{4}{\pi}\left\{ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots \right\} < \sum \frac{1}{n^2}$$

and the series $\sum \frac{1}{n^2}$ converges. Thus according to the above theorm this series converges uniformly. The plot of the first approximation curves of this series and of $f(x)$ are shown on plate VI. The following values were worked out for the approximation curves.

First approximation curve.

$$y = \frac{4}{\pi} \sin x$$

$x = 0$	$y = 0$
$x = 10°$	$y = .134$
$x = 20°$	$y = .268$
$x = 30°$	$y = .345$
$x = 40°$	$y = .505$
$x = 50°$	$y = .6083$
$x = 60°$	$y = .687$
$x = 70°$	$y = .742$
$x = 80°$	$y = .774$
$x = 90°$	$y = .74$

Second approximation curve.

$$y = \frac{4}{\pi}\left\{\sin x - \frac{\sin 3x}{3^2}\right\}$$

$x = 0$	$y = 0$
$x = 10°$	$y = .0948$
$x = 20°$	$y = .1975$
$x = 30°$	$y = .316$
$x = 40°$	$y = .4345$
$x = 50°$	$y = .5243$
$x = 60°$	$y = .6873$
$x = 70°$	$y = .7821$
$x = 80°$	$y = .8532$
$x = 90°$	$y = .8764$

Plotting $f(x)$, we have a straight line which passes through the origin

and increases to one at the point $x = \frac{\pi}{2}$ and then decreases to zero at $x = \pi$. For x negative we have the function $f(x)$ decreasing from $x = 0$ to the point $x = -\frac{\pi}{2}$ and then increasing to zero for the point $x = -\pi$. We can see from the plot, that the approximation curves approach $f(x)$ as n approaches infinity.

§3. Properties of uniform convergence

I *When an infinite series* is uniformly convergent in the neighborhood of a point of its convergency, the infinite series expresses a continuous function at this point.*[1]

Denoting the sum of the first $(n-1)$ terms of the series by $\sum_1^{n-1} u_n(x)$, we have

$$f(x) = \sum_1^{n-1} u_n(x) + R_n(x)$$

where $R_n(x)$ represents the remainder of the series after the $(n-1)$ term. Since the series is uniformly convergent in the neighborhood of x, a value (n) can be found such that $|R_n(x)|$ shall be less than $\frac{1}{3}\epsilon$, for any value from x to $x \pm h$, where ϵ is an arbitrarily small positive number. We know that,

$$|f(x \pm h) - f(x)| = \left|\sum_1^{n-1} u_n(x \pm h) - \sum_1^{n-1} u_n(x) + R_n(x \pm h) - R_n(x)\right|$$

---

[1] See Harnack's "Differential and Integral Calculus." page 220.

* "The terms of this series are here continuous functions of x.

We also know that the sum of the absolute values is less than or at most equal to the absolute value of the sum. Hence we can write

$$|f(x\pm h)-f(x)| \leqq \left|\sum_1^{n-1} u_n(x\pm h)-\sum_1^{n-1} u_n(x)\right| + |R_n(x\pm h)-R_n(x)|$$

$$|R_n(x)| < \tfrac{1}{3}\sigma$$

for (n) sufficiently large and as $|R_n(x\pm h)|$ can be made to differ from $|R_n(x)|$ by just as little as we please by the proper selection of h, we can make

$$|R_n(x\pm h)| < \tfrac{1}{3}\sigma.$$

Thus since we are dealing with absolute values we can say that,

$$|R_n(x\pm h)-R_n(x)| \leqq \tfrac{2}{3}\sigma$$

Thus we have

$$|f(x\pm h)-f(x)| \leqq \left|\sum_1^{n-1} u_n(x\pm h)-\sum_1^{n-1} u_n(x)\right| + \tfrac{2}{3}\sigma$$

But since the separate terms of the series are continous functions of x and $\sum_1^{n-1} u_n(x)$ is a sum of terms, finite in number, we can always choose a finite h so small that we may have

$$\left|\sum_1^{n-1} u_n(x \pm h) - \sum_1^{n-1} u_n(x)\right| < \tfrac{1}{3}\epsilon$$

Hence for any ϵ however small, a value h can be found for which

$$|f(x \pm h) - f(x)| < \epsilon$$

and our theorm is proved.

Example: Given,

$$f(x) = \frac{\sin x}{1^2} - \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} - \frac{\sin 4x}{4^2} + \cdots\cdots$$

where $\quad -\pi \leqq x \leqq \pi$

The absolute values of the series of maximum values

$$\sum_{k=1}^{\infty} |M_k| = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots$$

converges, because

$$\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots\right\} \leqq \sum_1^{\infty} \frac{1}{n^2}$$

and the series $\sum_1^{\infty} \frac{1}{n^2}$ converges. Thus according to theorm III, § 2, the given series converges uniformly. The series given above, that is $f(x)$, represents the broken line indicated on plate VII. We have then

$$\begin{aligned} fx &= x \quad \text{for } 0 \leqq x \leqq \tfrac{\pi}{2} \\ &= \pi - x \quad \text{“ } \tfrac{\pi}{2} \leqq x \leqq \pi \\ &= -x \quad \text{“ } -\tfrac{\pi}{2} \leqq x \leqq 0 \end{aligned}$$

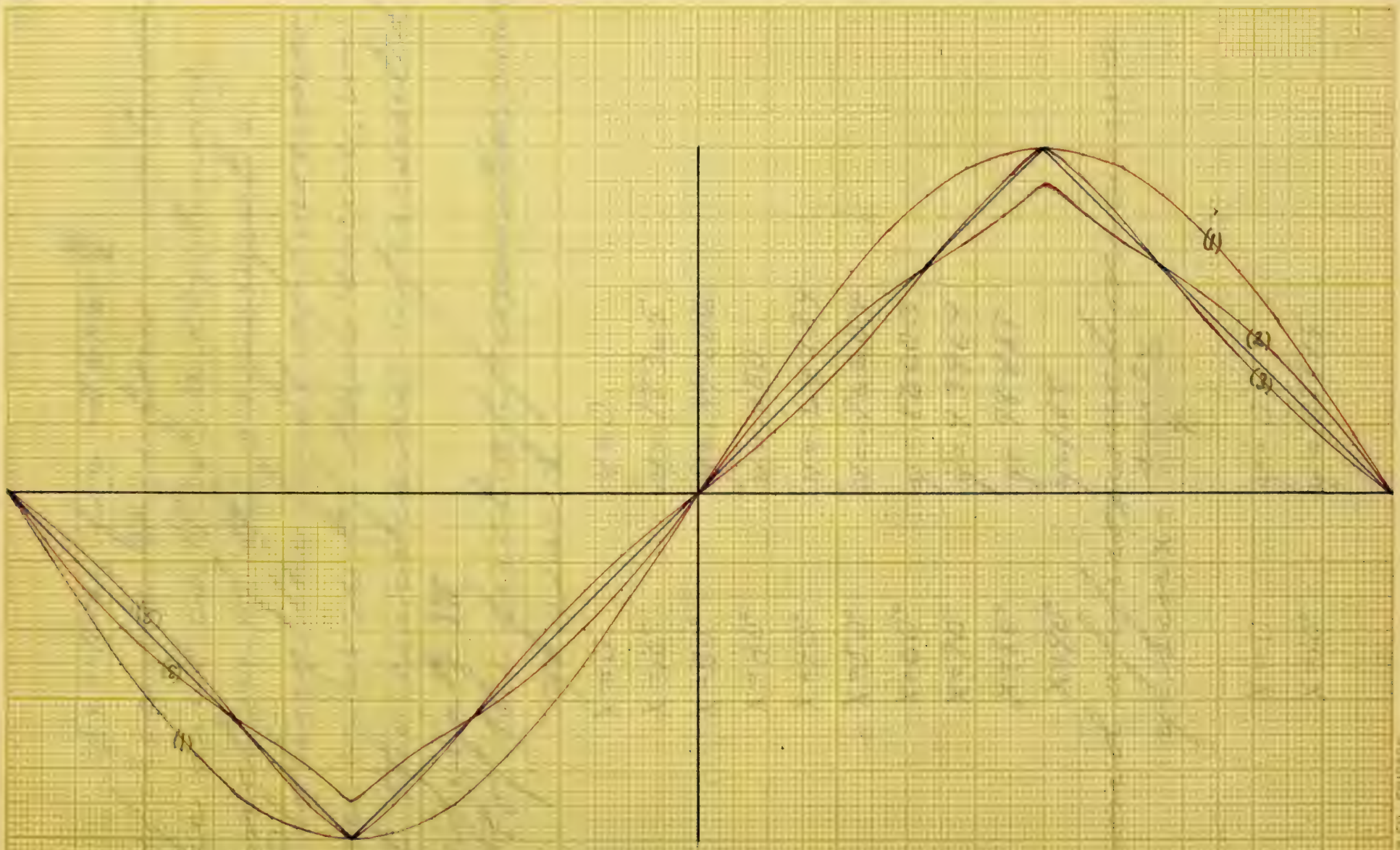


PLATE VII

$f x = -\pi - x$, for $-\pi \leqq x \leqq -\frac{\pi}{2}$

It represents therefore a continous line and is thus a continous function of the argument x."

Ploting the first approximation curves of this series, we get the curves as represented on Plate VII.

Plot of the first approximation curve

$y = \sin x$

$x = 0$	$y = 0$
$x = 10°$	$y = .17365$
$x = 20°$	$y = .34202$
$x = 30°$	$y = .500$
$x = 40°$	$y = .64279$
$x = 50°$	$y = .76604$
$x = 60°$	$y = .86603$
$x = 70°$	$y = .93969$
$x = 80°$	$y = .98481$
$x = 90°$	$y = 1.00$

Second approximation curve.

$y = \sin x - \frac{\sin 2x}{4}$

$x = 0$	$y = 0$
$x = 10°$	$y = .08815$

$X = 20°$	$y = .18133$
$X = 30°$	$y = .2835$
$X = 40°$	$y = .39654$
$X = 50°$	$y = .57484$
$X = 60°$	$y = .64953$
$X = 70°$	$y = .779$
$X = 80°$	$y = .89931$
$X = 90°$	$y = 1.00$

Third approximation curve.

$$y = \sin x - \frac{\sin 2x}{4} + \frac{\sin 3x}{9}$$

$X = 0$	$y = 0$
$X = 10°$	$y = .14365$
$X = 20°$	$y = .27755$
$X = 30°$	$y = .3946$
$X = 40°$	$y = .49281$
$X = 50°$	$y = .57534$
$X = 60°$	$y = .64453$
$X = 70°$	$y = .7235$
$X = 80°$	$y = .80309$
$X = 90°$	$y = .884$

We can see from the plate that the curves representing the approximation curves approach closer and closer to $f(x)$ as n increases

and our series is consequently uniformly convergent.

II. A uniformly convergent series can be integrated term by term in any finite interval.[1]

We can see from fig. 3, §1, that when $n \geq m$, where m is an arbitrarily large finite number,

$$\int_\alpha^\beta S_n(x)\,dx$$

will differ from $\int_\alpha^\beta f(x)\,dx$ by less than the area of the belt of breadth 2σ about the curve $f(x)$; i. e, by less than $2\sigma(\beta-\alpha)$; hence

$$\lim_{n=\infty} \int_\alpha^\beta S_n(x)\,dx = \int_\alpha^\beta f(x)\,dx$$

or we can prove this theorem as follows:

$$f(x) = \sum_1^{n-1} S_n(x) + R_n(x)$$

and thus we can write

$$\int_\alpha^\beta f(x)\,dx = \int_\alpha^\beta \sum_1^{n-1} S_n(x) + \int_\alpha^\beta R_n(x)\,dx.$$

---

1) "Uniform convergence and certain double limits," Prof. W. F. Osgood, Papers in Math. Bulletin, Nov. '96.

But we know that

$$\left|\int_\alpha^\beta R_n(x)\,dx\right| \leqq \int_\alpha^\beta |R_n(x)|\,dx < 2\delta(\beta-\alpha),$$

since $|R_n(x)| < 2\delta$, where $n \geqq m$.

Hence

$$\lim_{n=\infty} \int_\alpha^\beta R_n(x)\,dx = 0$$

and

$$\lim_{n=\infty} \int_\alpha^\beta \sum_1^{n-1} v_n(x)\,dx = \int_\alpha^\beta f(x)\,dx$$

and our theorem is proven.

Example: Given,

$$f(x) = \frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\cdots$$

The absolute values of the series of maximum values

$$\sum_{k=1}^{\infty} |M_k| = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots$$

converges, because

$$\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\cdots\right\} \leqq \sum_1^\infty \frac{1}{n^2}$$

and the series $\sum_1^\infty \frac{1}{n^2}$ converges.

Thus according to theorem III, § 2, the given series converges uniformly.

Ploting the first approximation curves we get the curves as represented on plate VIII.

Plot of the first approximation curve.

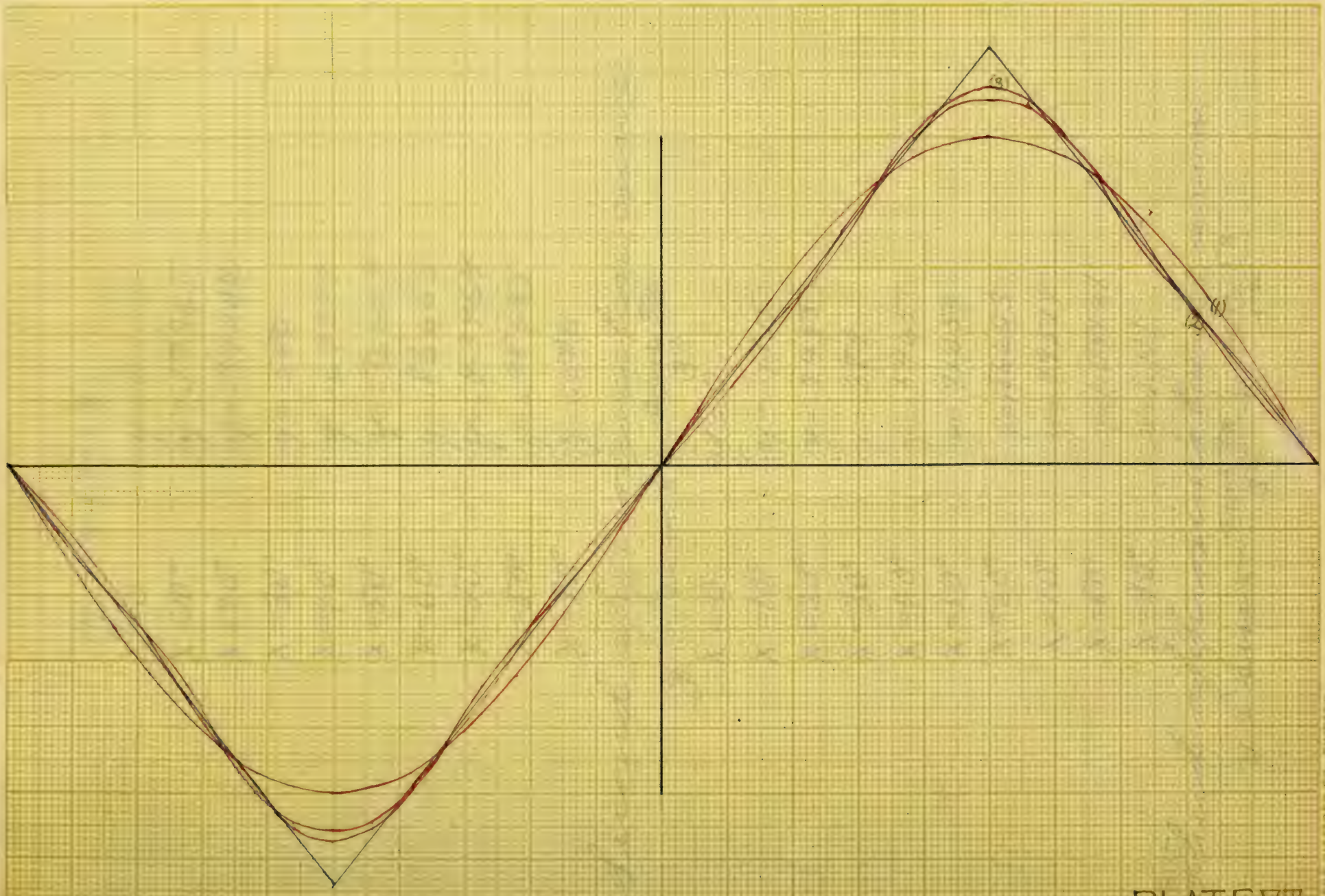


PLATE VIII

$y = \sin x$

$x = 0$	$y = 0$
$x = 10°$	$y = .17365$
$x = 20°$	$y = .34202$
$x = 30°$	$y = .500$
$x = 40°$	$y = .64279$
$x = 50°$	$y = .76604$
$x = 60°$	$y = .86603$
$x = 70°$	$y = .93964$
$x = 80°$	$y = .98481$
$x = 90°$	$y = 1.000$

Second approximation curve.

$$y = \sin x - \frac{\sin 3x}{3^2}$$

$x = 0$	$y = 0$
$x = 10°$	$y = .11813$
$x = 20°$	$y = .2458$
$x = 30°$	$y = .389$
$x = 40°$	$y = .54657$
$x = 50°$	$y = .71054$
$x = 60°$	$y = .86603$
$x = 70°$	$y = .99519$
$x = 80°$	$y = 1.09141$
$x = 90°$	$y = 1.111$

Third approximation curve.

$$y = \sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2}$$

$x=0$	$y=0$
$x=10°$	$y=.14879$
$x=20°$	$y=.28519$
$x=30°$	$y=.409$
$x=40°$	$y=.53289$
$x=50°$	$y=.67296$
$x=60°$	$y=.83139$
$x=70°$	$y=.98825$
$x=80°$	$y=1.11712$
$x=90°$	$y=1.151$

Ploting $f(x)$,

$x=0$	$y=0$
$x=90°$	$y=63.5 \text{ div.} = 1.270$

and we have a straight line between 0 and $\frac{\pi}{2}$ which increases and from $\frac{\pi}{2}$ to π we have $f(x)$, a straight line, which decreases from $y=63.5$ to 0. $f(x)$ is symmetrical in the first and third quadrant, except the values of x and y in the third quadrant are the negative of those in the first. We see from the plot that the higher the approximation curve, the nearer it approaches the

straight lines representing $f(x)$. Since this series is uniformly convergent it fulfills our condition for the integration of the series term by term.

III Let the series

$$u_1(x) + u_2(x) + u_3(x) + \ldots\ldots\ldots$$

converge for at least one value of x, say x_0, in the interval (α, β) and let $u_i(x)$ be continuous and have a continuous derivative $u_i'(x)$ in the interval; finally let the series of the derivatives

$$u_1'(x) + u_2'(x) + u_3'(x) + \ldots\ldots\ldots$$

converge uniformly. Then the u-series converges for all values of x in the interval (α, β); and if its value be denoted by $f(x)$, $f(x)$ has a continuous derivative, given by differentiating the u-series term by term,[11]

$$f'(x) = u_1'(x) + u_2'(x) + u_3'(x) + \ldots\ldots\ldots$$

---

(11) "Uniform convergence and certain double limits, Osgood, Math. Bulletin, Nov. '96.

Let us denote the value of the u_i-series by $\varphi(x)$, then

$$\varphi(x) = u_1'(x) + u_2'(x) + \cdots\cdots\cdots$$

Each of the terms of the series is continuous and thus $\varphi(x)$ is continuous and by theorm II, § 3,

$$\int_{x_0}^{x} \varphi(x)\,dx = \int_{x_0}^{x} u_1'(x)\,dx + \int_{x_0}^{x} u_2'(x)\,dx + \cdots\cdots$$

$$= [u_1(x) - u_1(x_0)] + [u_2(x) - u_2(x_0)] + \cdots\cdots$$

The series $u_1(x_0) + u_2(x_0) + \cdots$ is convergent, hence $u_1(x) + u_2(x) + \cdots$ is convergent and if its value be denoted by $f(x)$ then

$$\int_{x_0}^{x} \varphi(x)\,dx = f(x) - f(x_0)$$

The left hand side of this equation is a continuous function of x. Hence $f(x)$ is continuous and

$$f'(x) = \varphi(x)$$

and our theorm is proved.

Dini proves this theorm in the following manner.[1] We will let $\Sigma u_n'$, the series of the derivitives,

---

[1] See, "Dini Functionen Theorie," § 105. Law XI.

be convergent and let m' be such a number, that for $m \geqq m'$ and for all values of x between α and β the remainder $R'_m(x)$ of the series $\sum u'_m(x)$ is numerically less than $\frac{\sigma}{8}$. For $m > m'$ we have in absolute value

$$\sum_{m'+1}^{m} u'_m(x) < \frac{\sigma}{4},$$

because this series is uniformly convergent by hypothesis. We can put

$$\sum_{1}^{m}\left[\frac{u_m(x_0+\delta)-u_m(x_0)}{\delta} - u'_m(x_0)\right] = \sum_{1}^{m'}\left[\frac{u_n(x_0+\delta)-u_n(x_0)}{\delta} - u'_n(x_0)\right] + \sum_{m'+1}^{m}\left[u'_n(x_0+\theta\delta)-u'_n(x_0)\right]$$

in which $0 < \theta < 1$ and which is dependent respectively on m and δ and this is true for all terms of the sum $\sum_{m'+1}^{m}$ and thus we see that

$$\left|\sum_{m'+1}^{m}\left[u'_m(x_0+\theta\delta)-u'_m(x_0)\right]\right| < \frac{\sigma}{2}$$

for all values of $m > m'$. Since on the other hand m' is finite, then there always exists a number σ, such that for all values of δ, which are numerically less than σ, the inequality holds

$$\left| \sum_{1}^{m'} \left[\frac{u_n(x_0+\delta) - u_n(x_0)}{\delta} - u_n'(x_0) \right] \right| < \frac{\sigma}{2}$$

Thus one can say that

$$\left| \sum_{1}^{m} \left[\frac{u_n(x_0+\delta) - u_n(x_0)}{\delta} - u_n'(x_0) \right] \right| < \sigma$$

for all values of $m > m'$ and for sufficiently small δ's and σ. Since Σu_n converges in the interval (α, β), for every value of δ, a corresponding value of $m \geqq m'$ exists for which we have

$$\frac{R_m(x_0+\delta)}{\delta} < \sigma \text{ and } \frac{R_m(x_0)}{\delta} < \sigma$$

Thus our theorm is proven.

Example: Given

$$f(x) = \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots,$$

where $-1 < a \leqq x \leqq +1$

Differenciating the seperate terms of the above series we get

$$1 - x + x^2 - x^3 + x^4 - x^5 + \ldots\ldots\ldots$$

the terms of which are continuous functions, and if we take the series of absolute values

$$\sum_{k=1}^{\infty} |M_n| = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots\ldots$$

we have a convergent series with-

in the given interval

$$-1 < a \leqq x \leqq b < 1.$$

and thus the series

$$1 - x + x^2 - x^3 + x^4 - x^5 + \ldots\ldots\ldots$$

is uniformly convergent within the same interval. Hence our condition for the differenciation of a series term by term is fulfilled and thus,

$$f'(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots\ldots\ldots$$

where $f'(x)$ is the derivctive of $f(x)$.

The plot of this series is given on plate V.

§4. Condition that $f(x)$ is continuous.

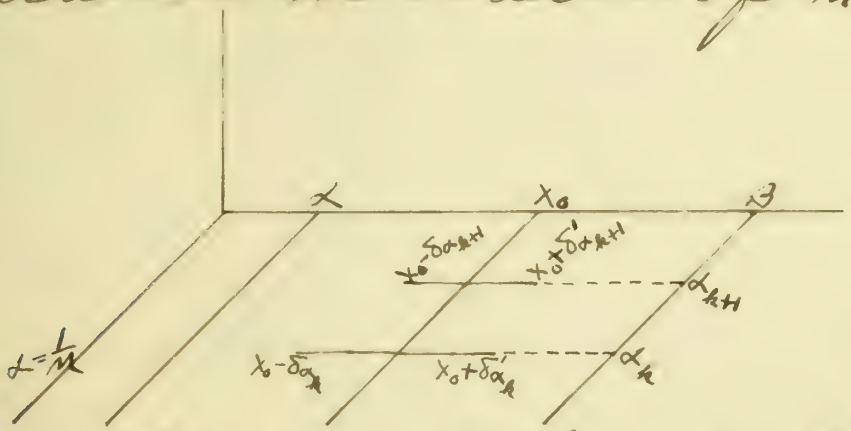
We know that uniform convergence gives us a sufficient but not a necessary condition that the function represented by a series having terms which are continuous functions, be itself continuous. We have seen that, if the double limit $\lim_{\substack{n=\infty \\ x=x_0}} S_n(x)$ exists and is equal to the value of the function for every point in the interval (α, β), then the series converges uniformly. But the series may be continuous even if the double limit does not exist at every point and equal to the function at that point.

Theorm: The necessary and sufficient condition that $f(x)$ should be a continuous function of x, at any point $x=x_0$ of the interval (α, β), is that the terms $u_n(x)$ of the series representing $f(x)$ be continuous and for $n = \alpha_1, \alpha_2 \cdots$, $(\alpha, \alpha_2 \cdots)$, being a set of points dense at zero)

we have about $x = x_0$ a definite interval greater than zero and which may vary with x_k, within which for every x, we have [1]

$$|f(x, x_k) - f(x)| < \varepsilon ,$$

where $f(x, x_k)$ is used as identically equal to the value of $\varphi_m(x)$ for $\frac{1}{m} = x_k$ [2]



Proof: We will first show that our condition is necessary. We have the function $f(x_0)$ continuous at the point x_0, hence from the condition for continuity, we have

---

(1) See, Prof. E. J. Townsend, "Paper given at Chicago 1900.

(2) "The same is to be understood by $f(x, x_k)$ throughout the entire discussion of this theorem."

(1) $|f(x_0) - f(x)| < \frac{\sigma}{3}$

where $x_0 - \bar{\delta}_0 \leqq x \leqq x_0 + \bar{\delta}_0'$

By hypothesis

$\lfloor_{n=\bar{\nu}} S_n(x) = f(x)$

and hence

$\lfloor_{\alpha_k=\bar{\nu}} f(x, \alpha_k) = f(x)$ for every constant value of x in (α, β) (α and β included). Hence for $x = x_0$, we have

(2) $|f(x_0 \alpha_k) - f(x_0)| < \frac{\sigma}{3}$

where $\alpha_0 < \alpha_k \leqq \alpha_s$

Since $f(x, \alpha_k)$ is continuous in x, we have

(3) $|f(x, \alpha_k) - f(x_0, \alpha_k)| < \frac{\sigma}{3}$

where $x_0 - \bar{\delta}_{\alpha_k} < x \leqq x_0 + \bar{\delta}'_{\alpha_k}$

Adding (1), (2) and (3) we have the condition

(4) $|f(x, \alpha_k) - f(x)| < \sigma$

for all values of x within the smaller of the two intervals $(x_0 - \bar{\delta}_0, x_0 + \bar{\delta}_0')$ and $(x_0 - \bar{\delta}_{\alpha_k}, x_0 + \bar{\delta}'_{\alpha_k})$ which proves the first part of our proposition. It is to be noticed however that the interval $(x_0 - \bar{\delta}_{\alpha_k}, x_0 + \bar{\delta}'_{\alpha_k})$ may vary with α_k and may have a lower limit zero and differs from uni-

form convergence in this respect.

Our condition is also sufficient; for we may show as follows that, when our condition exists for a set of points $\alpha_1, \alpha_2 \cdots \alpha_k \cdots$ it also exists for every value of n greater than a definite $n = n'$ from which the continuity of $f(x)$ at x_0 follows. We have then for every x within the interval $(x_0 - \delta_{\alpha_k}', x_0 + \delta_{\alpha_k}')$ the condition that,

$$|f(x) - f(x, \alpha_k)| < \delta \tag{5}$$

for every α_k from some definite value on. Since for a constant $x = x_0$ however, the limit

$$\lim_{n=\infty} S_n(x_0) = f(x_0)$$

and hence

$$\lim_{\alpha_k = 0} f(x_0 \alpha_k) = f(x_0),$$

we can select the α_k such that

$$|f(x_0, \alpha_k) - f(x_0, \alpha_{k+1})| < \delta, \tag{6}$$

where $\alpha_0 < \alpha_{k+1} < \alpha_k$

But for α_k and for α_{k+1}, the function $f(x, \alpha)$ is a continuous function of x alone, hence within definite limits for x, we have the two in-

equalities

(7) $|f(x,x_k) - f(x_0,x_k)| < \delta$

where $x_0 - \delta_{x_k} \leqq x \leqq x_0 + \delta'_{x_k}$

(8) $|f(x_0,x_{k+1}) - f(x,x_{k+1})| < \delta$

where $x_0 - \delta_{x_{k+1}} \leqq x \leqq x_0 + \delta'_{x_{k+1}}$

By adding (5),(6),(7) and (8) we have

(9) $|f(x) - f(x,x_{k+1})| < 4\delta$

for every x within the smaller of the two intervals

$$(x_0 - \delta_{x_k}, x_0 + \delta'_{x_k}), (x_0 - \delta_{x_{k+1}}, x_0 + \delta'_{x_{k+1}})$$

Hence our condition holds for every $n \geqq n'$ where n' is a definite value. Since the limit

$$\lim_{n=\infty} S_n(x) = f(x)$$

we have

(10) $|f(x_0,x_{k+1}) - f(x_0)| < \delta$,

if we select n' so that (7) and (10) are fulfilled. We have from (5)

(11) $|f(x,x_{k+1}) - f(x_0,x_{k+1})| < \delta$

and hence by addition of (9),(10) and (11) we have

$$|f(x) - f(x_0)| < 6\delta$$

for every x within a definite in-

terval differing from zero in extent. Hence $f(x)$ is continuous at x_0 and our theorem is demonstrated.

Example: Given

$$f(x) = (1-x) + x(1-x) + x^2(1-x) + \cdots\cdots + x^{n-1}(1-x) + \cdots\cdots$$

where $-1 \leqq x \leqq +1$.

Plotting the first approximation curve. Plate IX.

$$y = (1-x)$$

$x = -1$	$y = 2$
$x = -.8$	$y = 1.8$
$x = -.6$	$y = 1.6$
$x = -.5$	$y = 1.5$
$x = -.3$	$y = 1.3$
$x = -.2$	$y = 1.2$
$x = -.1$	$y = 1.1$
$x = 0$	$y = 1$
$x = .1$	$y = .9$
$x = .2$	$y = .8$
$x = .3$	$y = .7$
$x = .5$	$y = .5$
$x = .6$	$y = .4$
$x = .7$	$y = .3$
$x = 1$	$y = 0$

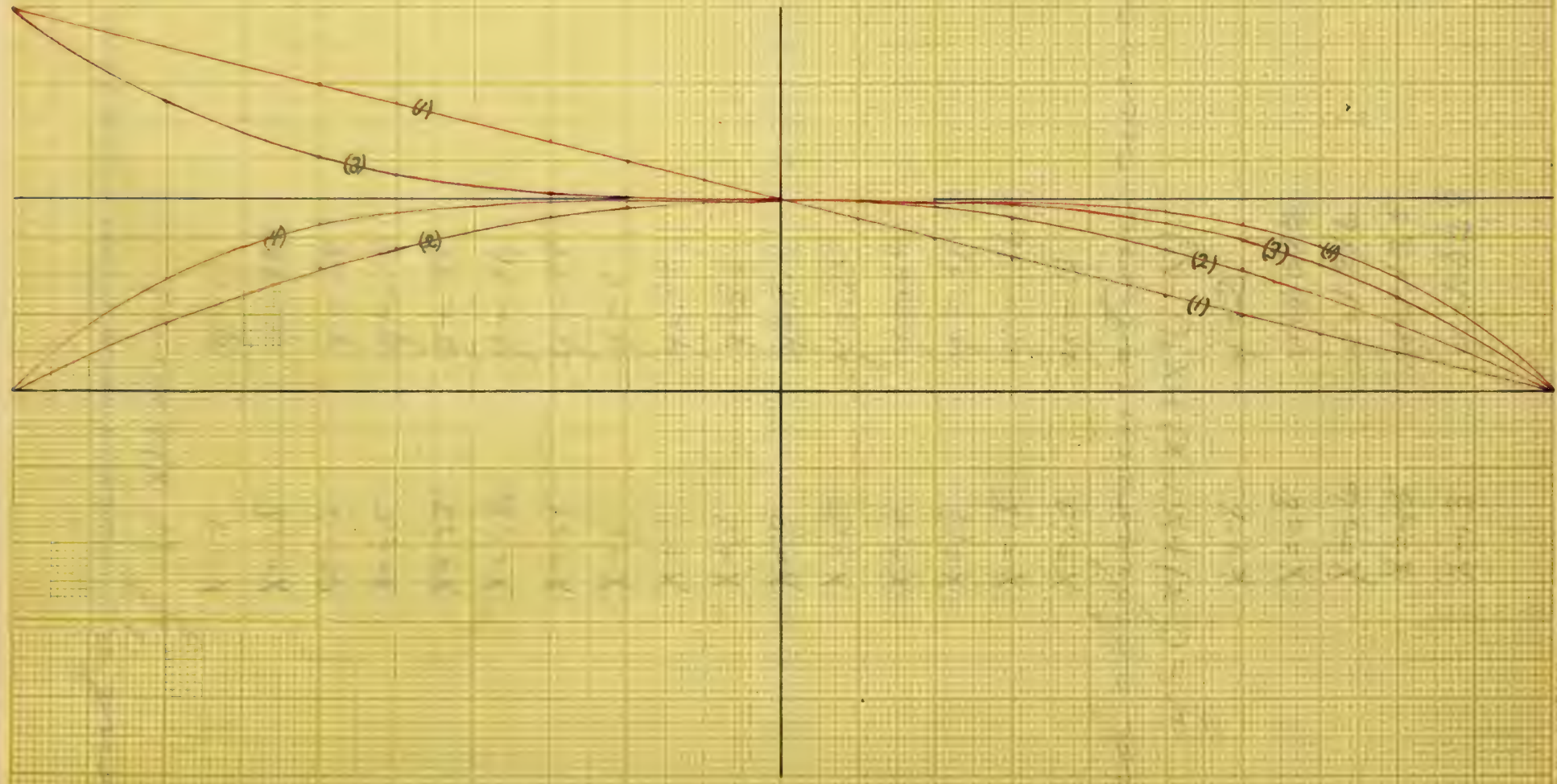
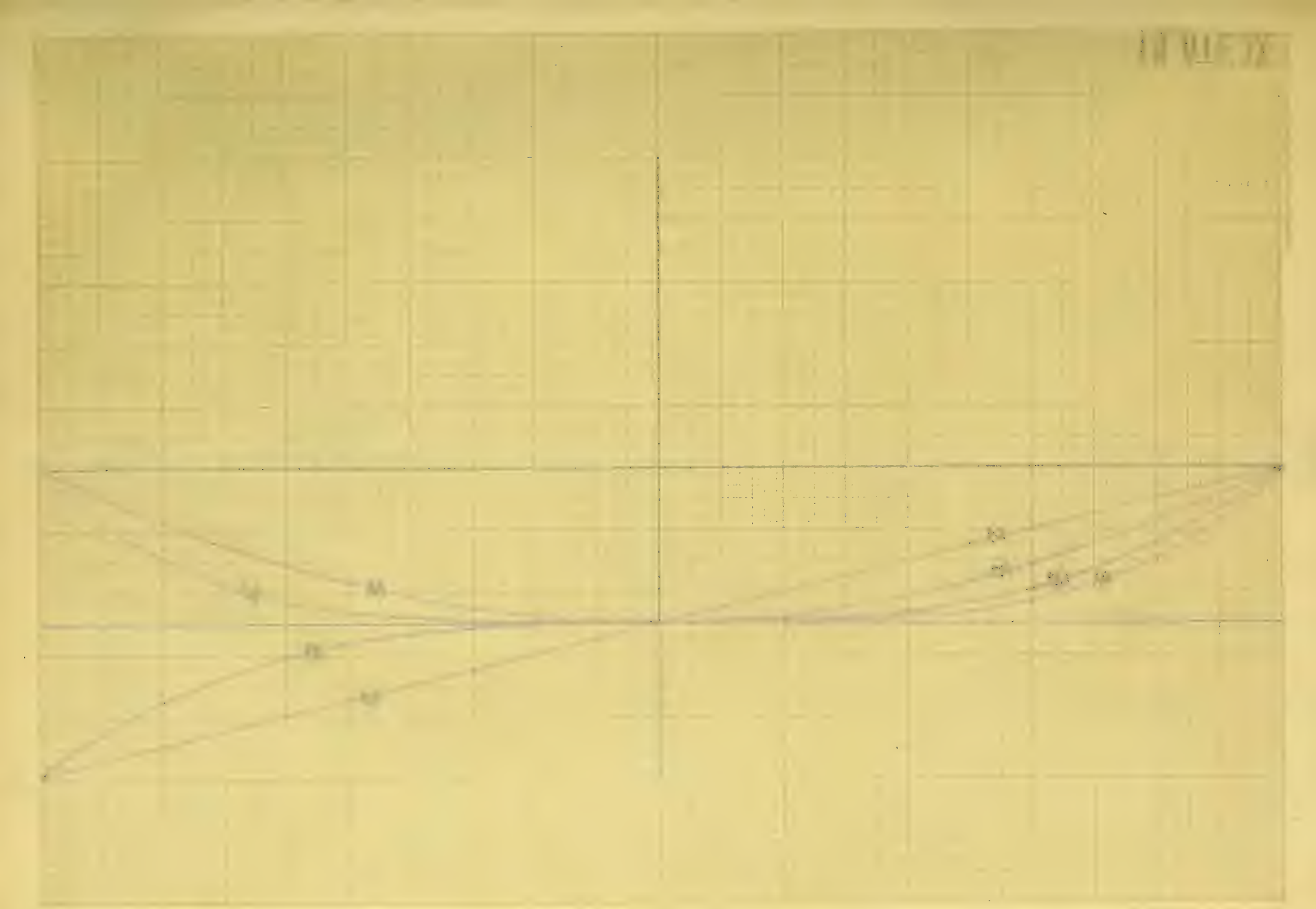


PLATE IX

EUGENE DIETZGEN CO., CHICAGO.



Second approximation curve

$y = (1-x) + x(1-x)$

$x = -1$	$y = 0$
$x = -.8$	$y = .36$
$x = -.6$	$y = .64$
$x = -.5$	$y = .75$
$x = -.3$	$y = .91$
$x = -.2$	$y = .96$
$x = -.1$	$y = .99$
$x = 0$	$y = 1$
$x = .1$	$y = .99$
$x = .2$	$y = .96$
$x = .3$	$y = .91$
$x = .5$	$y = .75$
$x = .6$	$y = .64$
$x = .7$	$y = .51$
$x = .8$	$y = .36$
$x = 1.0$	$y = 0$

Third approximation curve.

$y = (1-x) + x(1-x) + x^2(1-x)$

$x = -1$	$y = 2$
$x = -.8$	$y = 1.512$
$x = -.6$	$y = 1.216$
$x = -.5$	$y = 1.125$
$x = -.3$	$y = 1.027$

$X = -.2$	$y = 1.008$
$X = -.1$	$y = 1.001$
$X = 0$	$y = 1$
$X = .1$	$y = .999$
$X = .2$	$y = .992$
$X = .3$	$y = .973$
$X = .5$	$y = .875$
$X = .6$	$y = .784$
$X = .7$	$y = .657$
$X = .8$	$y = .488$
$X = 1$	$y = 0$

Fourth approximation curve.

$$y = (1+x) + x(1-x) + x^2(1-x) + x^3(1-x)$$

$X = -1$	$y = 0$
$X = -.8$	$y = .5904$
$X = -.6$	$y = .8704$
$X = -.5$	$y = .9375$
$X = -.3$	$y = .9919$
$X = -.2$	$y = .9984$
$X = -.1$	$y = .9999$
$X = 0$	$y = 1$
$X = .1$	$y = 9999$
$X = .2$	$y = .9984$
$X = .3$	$y = 9919$
$X = .5$	$y = 9375$

$X = .6 \qquad y = .8704$

$X = .7 \qquad y = .7599$

$X = .8 \qquad y = .5904$

$X = 1.0 \qquad y = 0$

$f(x) = \lim_{n=\infty} (1-x) + x(1-x) + \cdots + x^{n-1}(1-x) + \cdots = 1$

Hence $f(x)$ represents a straight line which is parallel to the x-axis and at a distance one from it. We see by the plot on plate IX that the approximation curves, at the points $x = \pm 1$, can never become equal to $f(x)$ at these points. Hence this series is now uniformly convergent in the interval $-1 \leqq X \leqq +1$. But since $f(x) = 1$ it must represent a continous function for all values of x in the given interval. Thus we see that the series may represent a continuous function but yet be now uniformly convergent in a given interval.